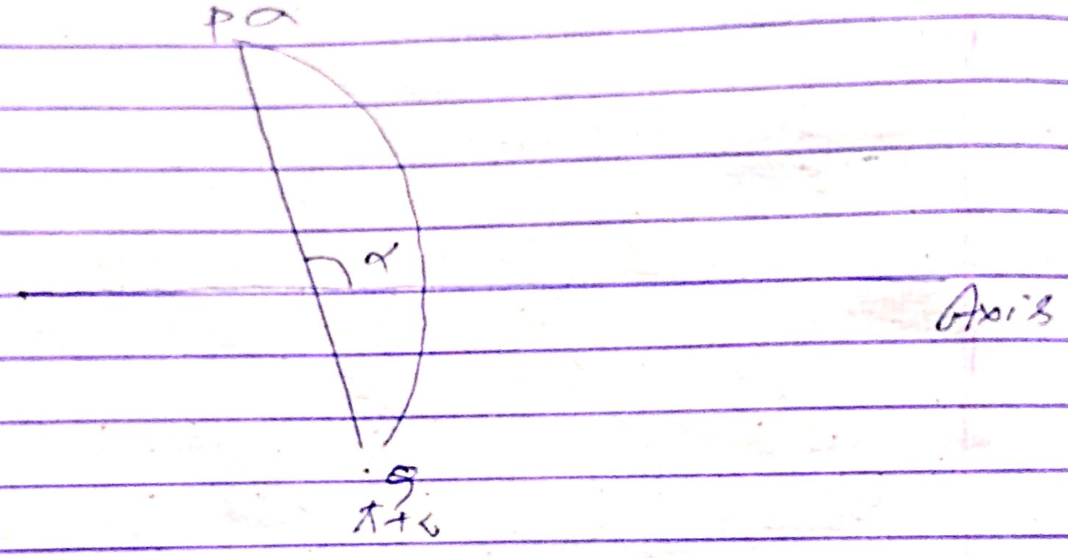
Name of College - S.S. College, J. [illegible]
Paper - Mathematics
Topic - Problem of polar equation of Conic Section (Cont'd)
Class - B.Sc I (Hons)
Time - 1:00 PM to 1:45 PM
Date - 04-09-2020
By - Samrendra Kumar

Q. Find the length of the focal chord of the conic $\frac{l}{r} = 1 + e\cos\theta$ which is inclined to the axis at an angle α. Deduce that the shortest focal chord of a conic is the latus-rectum.



Solution :→
Let the equation of the conic be $\frac{l}{r} = 1 + e\cos\theta$ ———— (1)

Here S = focus
Let PSQ be a focal chord of the conic.
Let α be the vectorial angle of P then $\pi+\alpha$ is the vectorial angle of Q.

Since the points $P(SP, \alpha)$ and $Q(SQ, \pi+\alpha)$ are on the conic (1)

Therefore

$$\frac{l}{SP} = 1 + e\cos\alpha$$

$$\Rightarrow SP = \frac{l}{1+e\cos\alpha} \quad \text{(II)}$$

and $$\frac{l}{SQ} = 1 + e\cos(\pi+\alpha)$$

$$\Rightarrow SQ = \frac{l}{1-e\cos\alpha} \quad \text{(III)}$$

Therefore, the length of the focal chord

$$PQ = SP + SQ$$

$$= \frac{l}{1+e\cos\alpha} + \frac{l}{1-e\cos\alpha}$$

$$= \frac{2l}{1-e^2\cos^2\alpha}$$ Which is the required length of the focal chord.

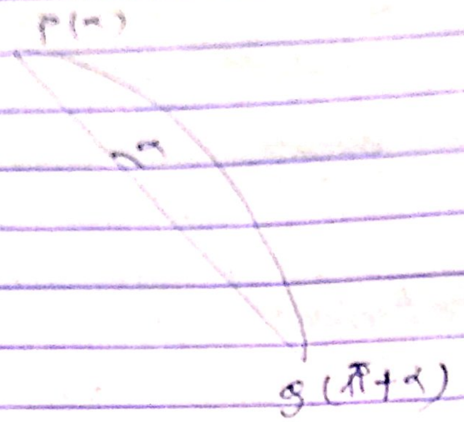
The focal chord PQ will be shortest when $1-e^2\cos^2\alpha$ is the greatest

The greatest value of $1-e^2\cos^2\alpha$ is 1

$\therefore 1 - e^2\cos^2\alpha = 1 \Rightarrow e^2\cos^2\alpha = 0$

$\cos\alpha = 0$ or $\alpha = \pi/2$

2. Prove that the locus of the middle points of focal chords of a conic section is a conic section.



Let the equation of the conic section be.

$$\frac{l}{r} = 1 + e\cos\theta \quad \text{——— (1)}$$

S is the focus of the conic section.

Let PSQ be a focal chord of the conic section.

Let α be vectorial angle of P. Then $\pi+\alpha$ is a vectorial angle of Q.

Since the points $P(SP, \alpha)$ and $Q(SQ, \pi+\alpha)$ are on the conic (1)

Therefore $\frac{l}{SP} = 1 + e\cos\alpha$

$$\frac{1}{SP} = \frac{1+e\cos\alpha}{l}$$

Similarly

$$\frac{1}{SQ} = \frac{1 - e\cos\alpha}{l}$$

Let R be the middle point of PSQ.

$$\therefore PR = RQ = \frac{1}{2} PQ$$

then R is (r_1, α)

Now $r_1 = SR = SP - PR = SP - \frac{1}{2} PQ$

$$2r_1 = 2SP - PQ$$

$$= 2SP - (SP + SQ)$$

$$= SP - SQ$$

$$= \frac{l}{1+e\cos\alpha} - \frac{l}{1-e\cos\alpha} = -\frac{2el\cos\alpha}{1-e^2\cos^2\alpha}$$

$$\Rightarrow r_1(1 - e^2\cos^2\alpha) + el\cos\alpha = 0$$

Therefore, the locus of $R(r_1, \alpha)$ is

$$r(1 - e^2\cos^2\theta) + el\cos\theta = 0$$

put $x = r\cos\theta$ and $y = r\sin\theta$

$$x^2 + y^2 = r^2$$

$$\therefore r\left(1 - \frac{e^2x^2}{r^2}\right) + el\frac{x}{r} = 0$$

$$\Rightarrow r^2 - e^2x^2 + elx = 0$$

$$\Rightarrow x^2 + y^2 - e^2x^2 + elx = 0$$

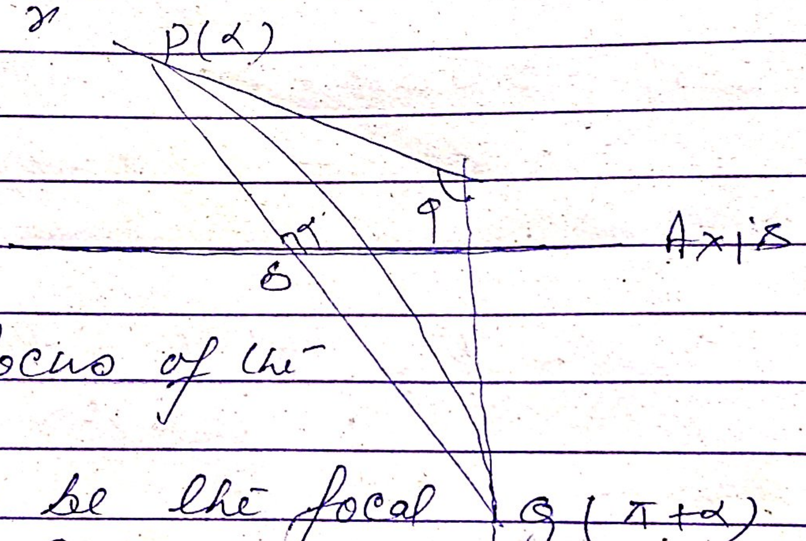
$$\Rightarrow (1-e^2)x^2 + y^2 + elx = 0$$ this represent a conic section

Problem: If a focal chord of the conic $\frac{l}{r} = 1 + e\cos\theta$ makes an angle α with the axis, show that the angle between the tangents at its extremities is

$$\tan^{-1}\left(\frac{2e\sin\alpha}{1-e^2}\right).$$

Solution:→ The equation of the conic is

$$\frac{l}{r} = 1 + e\cos\theta \quad \text{———} (1)$$



S is the focus of the conic.

Let PSQ be the focal chord of the conic.

Let α be the vectorial angle of P.

Then $\pi+\alpha$ is a vectorial angle of Q.

The equation of the tangent P(α) is

$$\frac{l}{r} = e\cos\theta + \cos(\theta-\alpha)$$

or $l = re\cos\theta + r(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$

But $x = r\cos\theta$, $y = r\sin\theta$

$l = ex + x\cos\alpha + y\sin\alpha$

or $l = x(e+\cos\alpha) + y\sin\alpha$.

The slope of this tangent

$= -\dfrac{\text{the co-efficient of } x}{\text{the co-efficient of } y}$

$= -\dfrac{e+\cos\alpha}{\sin\alpha}$

Similarly the slope of the tangent at $Q(\pi+\alpha)$

$= -\dfrac{e+\cos(\pi+\alpha)}{\sin(\pi+\alpha)} = -\dfrac{e-\cos\alpha}{-\sin\alpha}$

$= \dfrac{e-\cos\alpha}{\sin\alpha}$

If φ be the angle between the tangent at P and Q, then

$\tan\varphi = \dfrac{m_2-m_1}{1+m_1m_2} = \dfrac{\dfrac{e-\cos\alpha}{\sin\alpha} + \dfrac{e+\cos\alpha}{\sin\alpha}}{1+\left(\dfrac{e-\cos\alpha}{\sin\alpha}\right)\left(-\dfrac{e+\cos\alpha}{\sin\alpha}\right)}$

$= \dfrac{2e\sin\alpha}{\sin^2\alpha - e^2 + \cos^2\alpha} = \dfrac{2e\sin\alpha}{1-e^2}$

Hence $\varphi = \tan^{-1}\dfrac{2e\sin\alpha}{1-e^2}$, which is the required angle.